On the Degree Sequence of Random Geometric Digraphs

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Abstract

A random geometric digraph G_n is constructed by taking $\{X_1, X_2, \cdots X_n\}$ in $\mathbb{R}^2$ independently at random with a common bounded density function. Each vertex X_i is assigned at random a sector S_i of central angle α with inclination Y_i, in a circle of radius r (with vertex X_i as the origin). An arc is present from vertex X_i to X_j, if X_j falls in S_i. Suppose k is fixed and $\{k_n\}$ is a sequence with $1 \ll k_n \ll n^{1/2}$, as $n \to \infty$. We prove central limit theorems for $k-$ and k_n-nearest neighbor distance of out- and in-degrees in G_n. We also show that the degree distribution of this model, which varies with the probability distribution of the underlying point processes, can be either homogeneous or inhomogeneous. Our work should provide valuable insights for alternative mechanisms wrapped in real-world complex networks.

Keywords: Random geometric graph, Random scaled sector graph, Degree sequence, Central limit theorem, De-Poissonization.

1. Introduction

In random graph theory, degree sequences are among the most elementary and essential issues. The random geometric graphs $G(\mathcal{X}, r)$ have been well studied in the last decade, see the monograph [13], a short overview [18] and references therein. In order to investigate the typical vertex degree of $G(\mathcal{X}_n, r_n)$, Penrose([14]) defined an empirical process of k_n-nearest neighbor distances in $\mathcal{X}_n$, and showed the weak convergence of the finite-dimensional distributions of that process, scaled and centered, to a Gaussian limit process. He further considered the case $k_n = k$ fixed in [13] later. Given a finite point set $\mathcal{X} \in \mathbb{R}^d$ and given $x \in \mathcal{X}$, the k-nearest neighbor distance means the distance from x to its k-nearest neighbor in $\mathcal{X}$. In the geometric setting, the k-nearest neighbor distance is often a suitable vehicle to deal with degree-related properties of spatial point configurations[12]. It is also closely concerned with k-spacing in statistical testing, which has a number of applications, see the book [17], and is of interest in its own right.

In this paper we extend the method of Penrose and establish results analogous to the ones mentioned above for in-degree and out-degree of random geometric digraphs. Our result (Theorem 3) shows that the degree distribution of random geometric digraphs in the thermodynamic regime can be either homogeneous or inhomogeneous according to different underlying distributions of point processes. In particular, the degree distribution is Poisson-like when points are uniformly scattered, reminiscent of that of Erdös-Rényi random graphs, see the classic book [1](Chap.3); otherwise the degree distribution is highly skew (or inhomogeneous), similar with that of many large real-world graphs [7]. We also mention that the author was recently able to prove the maximum out/in-degrees are almost determined [19], and this phenomenon has been discovered in Erdös-Rényi random graphs [1]. For more discussions, see Section 2.1.

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Let $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$, $\{X_i\}$ are i. i. d. random variables in $\mathbb{R}^d$ with distribution F having a specified bounded density function f. Let $\mathcal{P}_n = \{X_1, X_2, \cdots, X_{N_n}\}$, $N_n \sim Poi(n)$. So $\mathcal{P}_n$ is a Poisson point process with intensity nf, coupled with $\mathcal{X}_n$. Let $\mathcal{H}_\lambda$ be a homogeneous Poisson process with intensity λ on $\mathbb{R}^d$ and $\|\cdot\|$ be l^2 norm on $\mathbb{R}^d$. Standard random geometric graphs $G(\mathcal{X}_n, r_n)$, $G(\mathcal{P}_n, r_n)$ are defined as in [13], that is, $G(\mathcal{X}_n, r_n)$ (or $G(\mathcal{P}_n, r_n)$) has vertex-set $\mathcal{X}_n$ (or $\mathcal{P}_n$) and an edge X_iX_j $(i \neq j)$ if $||X_i - X_j|| < r_n$. We always assume that $r_n \to 0$ as $n \to \infty$. We now define random geometric digraph models to use in this paper as follows:

Definition 1. $(d = 2)$ *Let $\alpha \in (0, 2\pi]$ be fixed. Let $\mathcal{Y}_n = \{Y_1, Y_2, \cdots, Y_n\}$ be i.i.d. random variables, taking values in $[0, 2\pi)$, with density function g. Associate every point $X_i \in \mathcal{X}_n$ a sector, which is centered at X_i, with radius r_n, amplitude α and elevation Y_i with respect to the x-axis horizontal direction anticlockwise. This sector is denoted as $S(X_i, Y_i, r_n)$. We denote by $G_\alpha(\mathcal{X}_n, \mathcal{Y}_n, r_n)$ (abbreviated as G_n) the digraph with vertex set $\mathcal{X}_n$, and with arc (X_i, X_j), $i \neq j$, present if and only if $X_j \in S(X_i, Y_i, r_n)$. We can define a Poisson version $G_\alpha(\mathcal{P}_n, \mathcal{Y}_{N_n}, r_n)$ (G'_n for short) similarly.*

In what follows, we will primarily take $g = \frac{1}{2\pi}1_{[0,2\pi)}$, that is, $Y_i \sim U[0, 2\pi)$. We will defer the discussion of the case of $d \geq 3$, general probability density function g and even other norms to Section 6. Actually, the above model has been first introduced in [4] under the name "random scaled sector graph", with $d = 2$, Euclidean norm and n points uniformly distributed in $[0, 1]^2$. This is an important variant of random geometric graph which has been revitalized recently in the context of wireless ad hoc networks, and it is used to analyze the performance of wireless sensor networks communicating through optical devices or directional antennae, which are significant in mobile communication[11]. Some basic properties and graph-theoretic parameters of this model have also been addressed[4, 5, 6], using basically combinatorial techniques and discretization.

The rest of this paper is organized as follows. Section 2 contains the statement of main results for $d = 2$, Y_i uniformly distributed. Section 3 discusses the asymptotic results for means and degree distribution. In Section 4, we give some moments preparatives for de-Poisson. Section 5 includes the proof of main theorems. Section 6 is devoted to higher dimension and general probability density function g.

2. Statement of main results

We will consider two asymptotic regimes. First, take $k_n \equiv k \in \mathbb{N}$. Second, let $k_n \to \infty$, and

$$\lim_{n\to\infty} \frac{k_n}{\sqrt{n}} = 0. \tag{1}$$

Notice that if we want the sequence $\{k_n\}_{n\geq 1}$ to converge as n tends to infinity, then the above two cases are only choices (and (1) is technically needed in the proofs). In the first regime, define $r_n = r_n(t)$ by $nr_n(t)^2 = t$, for $t > 0$, and in the second, define $r_n = r_n(t)$ by $nr_n(t)^2 = s(k_n + t\sqrt{k_n})$, for $s > 0$, $t \in \mathbb{R}$. Here we introduce a tunable parameter t to adjust the areas of sectors and t has nothing to do with "time", though we will study several random processes with t that evolves. Regulating t allows us to tackle the degree sequences in fine details. The reason why we choose such r_n is to ensure a non-degenerate limit, since nr_n^2 is a good measure of average degree, see the appendix A of [10]. We emphasize that k_n is a crucial parameter which appears in two respects, the scale on which the degree distribution tails are studied as well as the scaling for the radius r_n.

Before proceeding, we give some notations to ease statement. For $\lambda > 0$, let $\rho_\lambda(k) := P(Poi(\lambda) = k)$ and for $A \subseteq \mathbb{Z}^+$, let $\rho_\lambda(A) := P(Poi(\lambda) \in A)$. For $x \in \mathbb{R}^2$, let ϕ, Φ be

the density and distribution function of standard normal variables. Given $x \in \mathbb{R}^2$, define $B(x,r)$ the disk with center x and radius r, and let $B_n(x,t) := B(x, r_n(t))$, $S_n(x,y,t) := S(x,y,r_n(t))$ in both limit regimes. Following Penrose [13] we set $\mathcal{X}^x := \mathcal{X} \cup \{x\}$, if $\mathcal{X}$ is a finite set in $\mathbb{R}^2$ and $x \in \mathbb{R}^2$. Denote by $\#\mathcal{X}$ the number of elements in $\mathcal{X}$ and $\mathcal{X}(A) := \#(\mathcal{X} \cap A)$ for $A \subseteq \mathbb{R}^2$.

We will need some further definitions before we can state our main results. In the rest of the paper $f_{\max}$ will denote the essential supremum of the probability density function f, i.e. $f_{\max} := \sup\{u : |\{x : f(x) > u\}| > 0\}$. Here and in the rest of the paper $|\cdot|$ denotes Lebesgue measure. We assume $f_{\max} < \infty$ throughout the paper. Next, define the level set when $k_n \to \infty$ as $L_s := \{x \in \mathbb{R}^2 | sf(x) = \frac{2}{\alpha}\}$ and let $L_s^+ := \{x \in \mathbb{R}^2 | sf(x) > \frac{2}{\alpha}\}$. We also put a mild restriction on density function f: let $R := \big\{x \in \mathbb{R}^2 | f(x) > 0, \limsup\limits_{y \longrightarrow x} \frac{|f(y)-f(x)|}{\|y-x\|} < K\big\}$ with some $K < \infty$, and we always assume $F(R) = 1$. Let c, c' be various positive constants, and the values may change from line to line.

For Borel set $A \subseteq \mathbb{R}^2$, define $\xi_n^{out}(t,A)$, $\xi_n^{\prime out}(t,A)$ be the number of vertices in A of out-degrees at least k_n of G_n and G_n' respectively. More specifically,

$$\xi_n^{out}(t,A) = \sum_{i=1}^{n} 1_{[\mathcal{X}_n(S_n(X_i,Y_i,t)) \geq k_n+1] \bigcap [X_i \in A]}$$

$$\xi_n^{\prime out}(t,A) = \sum_{i=1}^{N_n} 1_{[\mathcal{P}_n(S_n(X_i,Y_i,t)) \geq k_n+1] \bigcap [X_i \in A]}$$

Similarly, for in-degree we have,

$$\xi_n^{in}(t,A) = \sum_{i=1}^{n} 1_{[\#\{X_j \in \mathcal{X}_n | X_i \in S_n(X_j,Y_j,t)\} \geq k_n+1] \bigcap [X_i \in A]}$$

$$\xi_n^{\prime in}(t,A) = \sum_{i=1}^{N_n} 1_{[\#\{X_j \in \mathcal{P}_n | X_i \in S_n(X_j,Y_j,t)\} \geq k_n+1] \bigcap [X_i \in A]}$$

Notice for the case $k_n \to \infty$, s is suppressed in the above expressions. Also, let $\xi_n^{out}(t) := \xi_n^{out}(t,\mathbb{R}^2)$ etc. for convenience.

The following two lemmas are intermediate steps to prove Theorem 1 and 2. We choose to state them without proof due to the limitation of space and they can be treated in parallel with Theorem 4.12 and 4.13 in [13] through a dependency graph argument.

Lemma 1. *Suppose that $k_n = k$ is fixed, and that A is a Borel set in $\mathbb{R}^2$. The finite-dimensional distributions of the process*

$$n^{-\frac{1}{2}}[\xi_n^{\prime out}(t,A) - E\xi_n^{\prime out}(t,A)] \quad , \quad t \geq 0$$

converge to those of a centered Gaussian process $(\xi_\infty^{\prime out}(t,A), t > 0)$ with covariance $E[\xi_\infty^{\prime out}(t,A)\xi_\infty^{\prime out}(u,A)]$ given by

$$\int_A \rho_{\frac{\alpha}{2}tf(x)}([k,\infty))f(x)\mathrm{d}x$$
$$+\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_A\int_{\mathbb{R}^2}\psi_\infty^{out}(z,f(x_1),y_1,y_2)f^2(x_1)\mathrm{d}z\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}y_2$$

with

$$\begin{aligned}\psi^{out}_\infty(z,\lambda,y_1,y_2) =& P(\{\mathcal{H}^z_\lambda(S(0,y_1,t^{\frac{1}{2}}))\geq k\}\cap\{\mathcal{H}^0_\lambda(S(z,y_2,u^{\frac{1}{2}}))\geq k\})\\ &-P(\mathcal{H}_\lambda(S(0,y_1,t^{\frac{1}{2}}))\geq k)P(\mathcal{H}_\lambda(S(z,y_2,u^{\frac{1}{2}}))\geq k)\end{aligned}$$

The finite-dimensional distributions of the process

$$n^{-\frac{1}{2}}[\xi'^{in}_n(t,A)-E\xi'^{in}_n(t,A)] \quad , \quad t\geq 0.$$

converge to those of a centered Gaussian process $(\xi'^{in}_\infty(t,A), t>0)$ *with covariance* $E[\xi'^{in}_\infty(t,A)\xi'^{in}_\infty(u,A)]$ *given by*

$$\int_A \rho_{\frac{\alpha}{2}tf(x)}([k,\infty))f(x)\mathrm{d}x+\int_A\int_{\mathbb{R}^2}\psi^{in}_\infty(z,\frac{\alpha}{2\pi}f(x_1))f^2(x_1)\mathrm{d}z\mathrm{d}x_1$$

with

$$\begin{aligned}\psi^{in}_\infty(z,\lambda) =& P(\{\mathcal{H}^z_\lambda(B(0,t^{\frac{1}{2}}))\geq k\}\cap\{\mathcal{H}^0_\lambda(B(z,u^{\frac{1}{2}}))\geq k\})\\ &-P(\mathcal{H}_\lambda(B(0,t^{\frac{1}{2}}))\geq k)P(\mathcal{H}_\lambda(B(z,u^{\frac{1}{2}}))\geq k)\end{aligned}$$

Let $\mathcal{W}$ denote homogeneous white noise of intensity π^{-1} on $\mathbb{R}^2$, that is, a centered Gaussian process indexed by bounded Borel sets in $\mathbb{R}^2$, with covariance $\mathrm{Cov}(\mathcal{W}(A),\mathcal{W}(B)) = \frac{1}{\pi}|A\cap B|$, where $|\cdot|$ as mentioned before is Lebesgue measure. Also, let $\mathcal{W}'$ denote homogeneous white noise of intensity $\frac{2}{\alpha}$.

Lemma 2. *Suppose that* $k_n\to\infty$, *that (1) holds, and that* A *is a Borel set in* $\mathbb{R}^2$. *Let* $s>0$ *and suppose* $F(A\cap L_s)>0$. *The finite-dimensional distributions of the process*

$$(nk_n)^{-\frac{1}{2}}[\xi'^{out}_n(t,A)-E\xi'^{out}_n(t,A)] \quad , \quad t\in\mathbb{R}$$

converge to those of a centered Gaussian process $(\xi'^{out}_\infty(t,A), t\in\mathbb{R})$ *with covariance* $E[\xi'^{out}_\infty(t,A)\xi'^{out}_\infty(u,A)]$ *given by*

$$\frac{|L_s\cap A|}{s(\pi\alpha)^2}\int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^2}\mathrm{Cov}(1_{[\mathcal{W}'(S(0,y_1,1))\leq t]},1_{[\mathcal{W}'(S(z,y_2,1))\leq u]})\mathrm{d}z\mathrm{d}y_1\mathrm{d}y_2$$

The finite-dimensional distributions of the process

$$(nk_n)^{-\frac{1}{2}}[\xi'^{in}_n(t,A)-E\xi'^{in}_n(t,A)] \quad , \quad t\in\mathbb{R}$$

converge to those of a centered Gaussian process $(\xi'^{in}_\infty(t,A), t\in\mathbb{R})$ *with covariance* $E[\xi'^{in}_\infty(t,A)\xi'^{in}_\infty(u,A)]$ *given by*

$$\frac{4\cdot|L_s\cap A|}{s\alpha^2}\int_{\mathbb{R}^2}\mathrm{Cov}(1_{[\mathcal{W}(B(0,1))\leq t]},1_{[\mathcal{W}(B(z,1))\leq u]})\mathrm{d}z.$$

Now we are ready to state our main results.

Theorem 1. *Suppose that* $k_n=k$ *is fixed. The finite-dimensional distributions of the process*

$$n^{-\frac{1}{2}}[\xi^{out}_n(t)-E\xi^{out}_n(t)] \quad , \quad t\geq 0$$

converge to those of a centered Gaussian process $(\xi^{out}_\infty(t), t>0)$ *with*

$$E[\xi^{out}_\infty(t)\xi^{out}_\infty(u)] = E[\xi'^{out}_\infty(t)\xi'^{out}_\infty(u)]-h(t)h(u),$$

where

$$h(t) = \int_{\mathbb{R}^2} \left\{ \rho_{\frac{\alpha}{2}tf(x)}(k-1)\frac{\alpha}{2}tf(x) + \rho_{\frac{\alpha}{2}tf(x)}([k,\infty)) \right\} f(x)\mathrm{d}x \tag{2}$$

The above result also holds in the case where the superscripts 'out' are replaced by 'in' everywhere.

Theorem 2. *Suppose that $k_n \to \infty$, and (1) holds. Let $s > 0$ and suppose $F(L_s) > 0$. The finite-dimensional distributions of the process*

$$(nk_n)^{-\frac{1}{2}}[\xi_n^{out}(t) - E\xi_n^{out}(t)] \quad , \quad t \in \mathbb{R}$$

converge to those of a centered Gaussian process $(\xi_\infty^{out}(t), t \in \mathbb{R})$ with

$$E[\xi_\infty^{out}(t)\xi_\infty^{out}(u)] = E[\xi_\infty^{\prime out}(t)\xi_\infty^{\prime out}(u)] - g(t)g(u),$$

where $g(t) = \phi(t)F(L_s)$.
The above result also holds in the case where the superscripts 'out' are replaced by 'in' everywhere.

To deal with the degree distribution, let $\eta_n^{out}(t, A)$ and $\eta_n^{in}(t, A)$ be the number of vertices in A of out-degree and in-degree k fixed in G_n respectively.

Theorem 3. *Suppose A is a Borel set in $\mathbb{R}^2$ and $\alpha \geq \pi$. If either $k_n = k$ fixed, or $k_n \to \infty$ and $n^{-1}k_n^2 \ln n \to 0$, then*

$$\lim_{n\to\infty} n^{-1}\xi_n^{out}(t, A) - E[n^{-1}\xi_n^{out}(t, A)] = 0 \qquad a.e. \tag{3}$$

Moreover,

$$\lim_{n\to\infty} n^{-1}\eta_n^{out}(t, A) = \int_A \rho_{\frac{\alpha}{2}tf(x)}(k)f(x)\mathrm{d}x \qquad a.e. \tag{4}$$

The above result also holds in the case where the superscripts 'out' are replaced by 'in' everywhere.

2.1 Discussion of Theorem 3.

We take expectation on both sides of (4), and let $p(k) := E\lim_{n\to\infty} n^{-1}\eta_n^{out}(t, \mathbb{R}^2)$, so the out-/in-degree distribution of $G_\alpha(\mathcal{X}_n, \mathcal{Y}_n, r_n(t))$, where $nr_n(t)^2 = t$, is

$$p(k) = \frac{(\frac{\alpha}{2}t)^k}{k!}\int_{\mathbb{R}^2} e^{-\frac{\alpha}{2}tf(x)}f(x)^{k+1}\mathrm{d}x, \qquad k \in \mathbb{N}\cup\{0\} \tag{5}$$

If we take the uniform density function $f(x) = 1_{[0,1]^2}(x)$ in (5), then we see that $p(k) = e^{-\frac{\alpha}{2}t}(\frac{\alpha}{2}t)^k/k!$, $k \geq 0$; that is, the degree distribution is $Poi(\frac{\alpha}{2}t)$.

If we take the standard multivariate normal density function $f(x) := f(x_1, x_2) = (1/2\pi)e^{-(x_1^2+x_2^2)/2}$, then through the polar coordinate transformation and integration by parts, we obtain $p(k) = (4\pi/\alpha t) - e^{-\alpha t/4\pi}\sum_{i=0}^k(\alpha t/4\pi)^{i-1}/i!$, $k \geq 0$. It is easy to see that $p(k) \to 0$ as $k \to \infty$; and furthermore, since $p(0) = (4\pi/\alpha t)(1 - e^{-\alpha t/4\pi})$, $p(0) \to 1$ as $t \to 0$ and $p(0) \to 0$ as $t \to \infty$. These observations allow us presumably adjust the parameter t to get different skew degree distributions especially for small k. However, the degree distribution in (5) has a light tail in contrast to the power law distributions [7] because of the fast decay as k tends to infinity. To be precise, by (5) and Stirling formula,

$$p(k) \leq \frac{(\frac{\alpha}{2}tf_{\max})^k}{k!}\int_{\mathbb{R}^2} f(x)\mathrm{d}x = (1 + o(1)) \cdot \frac{(\alpha tef_{\max})^k}{(2k)^k\sqrt{2\pi k}} \ll k^{-\beta}$$

for any $\beta > 0$ as $k \to \infty$.

On the other hand, if we want to find a suitable density function f for a given probability distribution $p(k)$ satisfying $p(k) \geq 0$ and $\sum_{k=0}^{\infty} p(k) = 1$, then we simply solve the equation (5), which is the first kind nonlinear singular Fredholm integral equation [3]. However, only approximation solutions of this kind of equations may be obtained by using iterative methods and the existence of solution is not known in general.

3. Proof of means and degree distribution

Proposition 1. *(out-degree) Suppose $A \subseteq \mathbb{R}^2$ is a Borel set. If $k_n = k$ is fixed, then*

$$\lim_{n\to\infty} n^{-1} E[\xi_n^{out}(t, A)] = \int_A \rho_{\frac{\alpha}{2}tf(x)}([k, \infty))f(x)\mathrm{d}x \tag{6}$$

If $k_n \to \infty$, and (1) holds, then

$$\lim_{n\to\infty} n^{-1} E[\xi_n^{out}(t, A)] = F(L_s^+ \cap A) + \Phi(t)F(L_s \cap A) \tag{7}$$

Proof. Let $p_n(x, y, t) = F(S_n(x, y, t))$. Then

$$E[\xi_n^{out}(t, A)] = \frac{n}{2\pi} \int_0^{2\pi} \int_A P[Bin(n-1, p_n(x, y, t)) \geq k_n]f(x)\mathrm{d}x\mathrm{d}y \tag{8}$$

Suppose k_n is fixed, and $x \in R$, then f is continuous at x and $np_n(x, y, t) \to \frac{\alpha}{2}tf(x)$ by mean-value theorem of integrals. Therefore $P[Bin(n-1, p_n(x, y, t)) \geq k]$ tends to $\rho_{\frac{\alpha}{2}tf(x)}([k, \infty))$. Then (6) holds by (8) and dominated convergence theorem.

Suppose $k_n \to \infty$, (1) holds and $x \in R$, then $np_n(x, y, t) \sim n\frac{\alpha}{2}r_n^2 f(x) \sim s\frac{\alpha}{2}f(x)k_n$, and by Chernoff bounds (see e.g.[8]), $P[Bin(n-1, p_n(x, y, t)) \geq k_n]$ tends to 1, if $sf(x) > \frac{2}{\alpha}$; and tends to 0, if $sf(x) < \frac{2}{\alpha}$. Then for $x \in R \cap L_s$,

$$\begin{aligned} np_n(x, y, t) &= n\frac{\alpha}{2}r_n^2 f(x) + n\int_{S_n(x,y,t)} (f(z) - f(x))\mathrm{d}z \\ &= k_n + tk_n^{\frac{1}{2}} + \Theta(n(k_n/n)^{3/2}) \end{aligned}$$

Hence, by (1),

$$np_n(x, y, t) = k_n + tk_n^{\frac{1}{2}} + o(k_n^{\frac{1}{2}}), \quad x \in R \cap L_s \tag{9}$$

Then let $p_n = p_n(x, y, t)$, by DeMoivre-Laplace limit theorem and (9), we have

$$\begin{aligned} &P[Bin(n-1, p_n) \geq k_n] \\ &\quad = P\left[\frac{Bin(n-1, p_n) - EBin(n-1, p_n)}{\sqrt{np_n}} \geq \frac{k_n - (n-1)p_n}{\sqrt{np_n}}\right] \to \Phi(t) \end{aligned}$$

So (7) follows from (8) by dominated convergence theorem. □

Proposition 2. *(in-degree) The same results hold when replace superscripts "out" by "in" in Proposition 1.*

Proof. Let $q_n(x, t) = \frac{\alpha}{2\pi} \cdot F(B_n(x, t))$. Then

$$E[\xi_n^{in}(t, A)] = n\int_A P[Bin(n-1, q_n(x, t)) \geq k_n]f(x)\mathrm{d}x.$$

From Palm theory, similarly we have

$$E[\xi_n^{\prime in}(t,A)] = n\int_A P[Poi(nq_n(x,t)) \geq k_n] f(x)\mathrm{d}x.$$

The remain proof is in a similar spirit with that of Proposition 1. Hence we omit it. □

We remark here that Proposition 1 and 2 still hold for corresponding Poisson case.

Proof of Theorem 3. Define a σ filtration: $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and for $1 \leq i \leq n$, $\mathcal{F}_i = \sigma\{(X_1,Y_1),(X_2,Y_2),\cdots,(X_i,Y_i)\}$.

For out-degree, $\xi_n^{out}(t,A) - E[\xi_n^{out}(t,A)] = \sum_{i=1}^n M_{i,n}^{out}$, with $M_{i,n}^{out} = E[\xi_n^{out}(t,A)|\mathcal{F}_i] - E[\xi_n^{out}(t,A)|\mathcal{F}_{i-1}]$. Let $\xi_{n,i}^{out}(t,A)$ be the number of vertices in A of $G(\mathcal{X}_{n+1}\backslash\{X_i\}, \mathcal{Y}_{n+1}\backslash\{Y_i\}, r_n)$ having out-degree at least k_n. Thereby, $M_{i,n}^{out} = E[\xi_n^{out}(t,A) - \xi_{n,i}^{out}(t,A)|\mathcal{F}_i]$.

We now claim that: For finite set $\mathcal{X} \subseteq \mathbb{R}^2$ and $x \in \mathcal{X}$, there are at most $8k$ points $z \in \mathcal{X}$ having x as their $(\leq k)-th$ nearest neighbor, for any $k \in \mathbb{N}$. Here x is the $k-th$ nearest neighbor of z in $\mathcal{X}$ means if we order quantities $\{||w-z|| : w \in \mathcal{X}\backslash\{z\}\}$ increasingly, then $||x-z||$ will be the $k-th$ item in this sequence. Proof. We take a cone with vertex x, central angle $\pi/4$. It's easy to see that there are at most k_n points of $\mathcal{X}$ having x as their $(\leq k)-th$ nearest neighbor, since we may look for these points from near to far. The claim follows since the plane is covered by 8 such cones.

Therefore,

$$\begin{aligned} |\xi_n^{out}(t,A) - \xi_{n,i}^{out}(t,A)| &\leq |\xi_n^{out}(t,A) - \tilde{\xi}_{n+1}^{out}(t,A)| + |\tilde{\xi}_{n+1}^{out}(t,A) - \xi_{n,i}^{out}(t,A)| \\ &\leq (8k_n+1) + (8k_n+1) \leq 18k_n, \end{aligned}$$

where let $\tilde{\xi}_{n+1}^{out}(t,A)$ denote the number of vertices in A of out-degrees at least k_n of $G(\mathcal{X}_{n+1}, \mathcal{Y}_{n+1}, r_n)$. Then $|M_{i,n}^{out}| \leq 18k_n$. For $\varepsilon > 0$, by Azuma inequality, see e.g.[2],

$$P[|\xi_n^{out}(t,A) - E[\xi_n^{out}(t,A)]| > \varepsilon n] \leq 2e^{-\varepsilon^2 n^2/648nk_n^2}.$$

By Borel-Cantelli Lemma, (3) follows. The in-degree case can be proved similarly.

To prove (4), we notice

$$\eta_n^{out}(t,A) = \sum_{i=1}^n 1_{[\mathcal{X}_n(S_n(X_i,Y_i,t)) \geq k_n+1]\bigcap[X_i \in A]} - \sum_{i=1}^n 1_{[\mathcal{X}_n(S_n(X_i,Y_i,t)) \geq k_n+2]\bigcap[X_i \in A]}$$

and by (3) and the proof of Proposition 1, the result follows immediately. The in-degree case also follows similarly. □

4. Some moments for de-Poissonization

In this section we will develop some moments for non-Poisson case in the limit regime $k_n \to \infty$, which is crucial to de-Poisson Lemma 1 and 2.

For $n, m \in \mathbb{N}$, set

$$T_{m,n}^{out}(t) := \sum_{i=1}^m 1_{[\mathcal{X}_m(S_n(X_i,Y_i,t)\backslash\{X_i\}) \geq k_n]}$$

and

$$T_{m,n}^{in}(t) := \sum_{i=1}^m 1_{[\#\{X_j \in \mathcal{X}_m \backslash\{X_i\} | X_i \in S_n(X_j,Y_j,t)\} \geq k_n]}$$

Then we see $T^{out}_{n,n}(t) = \xi^{out}_n(t), T^{out}_{N_n,n}(t) = \xi'^{out}_n(t)$ and $T^{in}_{n,n}(t) = \xi^{in}_n(t), T^{in}_{N_n,n}(t) = \xi'^{in}_n(t)$. Set $\tilde{D}^{out}_{m,n}(t) := T^{out}_{m+1,n}(t) - T^{out}_{m,n}(t)$, then $\tilde{D}^{out}_{m,n}(t) = D^{out}_{m,n}(t) + \hat{D}^{out}_{m,n}(t)$, where

$$D^{out}_{m,n}(t) = \sum_{i=1}^{m} 1_{[\mathcal{X}_m(S_n(X_i,Y_i,t)\backslash\{X_i\})=k_n-1]\cap[X_{m+1}\in S_n(X_i,Y_i,t)]}$$

$$\hat{D}^{out}_{m,n}(t) = 1_{[\mathcal{X}_m(S_n(X_{m+1},Y_{m+1},t))\geq k_n]}$$

Set $\tilde{D}^{in}_{m,n}(t) := T^{in}_{m+1,n}(t) - T^{in}_{m,n}(t)$, then $\tilde{D}^{in}_{m,n}(t) = D^{in}_{m,n}(t) + \hat{D}^{in}_{m,n}(t)$, where

$$D^{in}_{m,n}(t) = \sum_{i=1}^{m} 1_{[\#\{X_j\in\mathcal{X}_m\backslash\{X_i\}|X_i\in S_n(X_j,Y_j,t)\}=k_n-1]\cap[X_i\in S_n(X_{m+1},Y_{m+1},t)]}$$

$$\hat{D}^{in}_{m,n}(t) = 1_{[\#\{X_j\in\mathcal{X}_m|X_{m+1}\in S_n(X_j,Y_j,t)\}\geq k_n]}$$

We denote binomial probability $\beta_{n,p}(k) := P(Bin(n,p) = k)$. The next lemma will be repeatedly used in this section, see [13, 14].

Lemma 3. *(a) Suppose $n, k \in \mathbb{N}$ with $k < n$. Then $\beta_{n,p}(k)$ is maximized over $p \in (0,1)$ by setting $p = k/n$, and $p\beta_{n,p}(k)$ is maximized over $p \in (0,1)$ by setting $p = (k+1)/(n+1)$.*

(b) Suppose $\{j_n\}_{n\geq 1}$ is a sequence of integers satisfying $j_n \to \infty$ and $(j_n/n) \to 0$ as $n \to \infty$. Suppose $t \in \mathbb{R}$ and $\{p_n\}_{n\geq 1}$ is a sequence in $(0,1)$ satisfying $(j_n - np_n)/(np_n)^{1/2} \to t$ as $n \to \infty$. Then

$$j_n^{1/2}\beta_{n,p_n}(j_n) \to \phi(t) \qquad \text{as } n \to \infty.$$

Lemma 4. *Suppose $k_n \to \infty$ and (1) holds. Then*

$$\lim_{n\to\infty} \sup_{\{m||m-n|\leq n^{2/3}\}} |k_n^{-1/2} E\tilde{D}^{out}_{m,n}(t) - \phi(t)F(L_s)| = 0$$

The same formula holds when replace superscript "out" by "in".

Proof. Take $\{m_n\}_{n\geq 1}$ with $|m_n - n| \leq n^{2/3}$.

For out-degree, we have

$$\begin{aligned} k_n^{-1/2} ED^{out}_{m_n,n}(t) \ &= \ \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^2} m_n k_n^{-1/2} P(\mathcal{X}_{m_n-1}(S_n(x,y,t)) = k_n - 1) \\ &\quad \cdot F(S_n(x,y,t))F(\mathrm{d}x)\mathrm{d}y. \end{aligned} \tag{10}$$

Let $x \in R \cap L_s$, then $\mathcal{X}_{m_n-1}(S_n(x,y,t))$ is binomial with parameters $m_n - 1$ and $F(S_n(x,y,t))$, and by (9), (1) the mean is

$$\begin{aligned} m_n F(S_n(x,y,t)) \ &= \ (1 + O(n^{-1/3}))(k_n + tk_n^{1/2} + o(k_n^{1/2})) \\ &= \ k_n + tk_n^{1/2} + o(k_n^{1/2}), \qquad x \in R \cap L_s \end{aligned} \tag{11}$$

By Lemma 3,

$$\lim_{n\to\infty} k_n^{1/2} P(\mathcal{X}_{m_n-1}(S_n(x,y,t)) = k_n - 1)) = \phi(t), \quad x \in R \cap L_s$$

Also, by Chernoff bounds and Proposition 1,

$$\lim_{n\to\infty} k_n^{1/2} P(\mathcal{X}_{m_n-1}(S_n(x,y,t)) = k_n - 1)) = 0, \quad x \in R\backslash L_s$$

Hence for $x \in R$, the integrand on the right hand side of (10) tends to $\frac{1}{2\pi}\phi(t)1_{L_s}(x)$. Also, by Lemma 3, $(m_n/k_n)F(S_n(x,y,t))$ and $k_n^{1/2}\sup_{0<p<1}\beta_{m_n-1,p}(k_n-1)$ are uniformly bounded. So, $k_n^{-1/2}ED^{out}_{m_n,n}(t)$ tends to $\phi(t)F(L_s)$ by dominated convergence theorem. Since $0 \le \hat{D}^{out}_{m_n,n}(t) \le 1$, $k_n^{-1/2}E\hat{D}^{out}_{m_n,n}(t)$ tends to 0. The first part of the lemma then follows.

For in-degree, we first introduce some notations. Let $\tilde{f} := \frac{\alpha}{2\pi}f$, and for Borel set $A \subseteq \mathbb{R}^2$, let $\tilde{\mathcal{X}}_n(A) \sim Bin(n, \tilde{F}(A))$, where $\tilde{F}(A) := \int_A \tilde{f}(x)\mathrm{d}x$.

Consequently, we have

$$k_n^{-1/2}ED^{in}_{m_n,n}(t) = \int_{\mathbb{R}^2} m_n k_n^{-1/2} P(\tilde{\mathcal{X}}_{m_n-1}(B_n(x,t)) = k_n - 1)\tilde{F}(B_n(x,t))F(\mathrm{d}x).$$

Let $x \in R \cap L_s$, as mentioned above, $\tilde{\mathcal{X}}_{m_n-1}(B_n(x,t))$ is binomial with parameters $m_n - 1$ and $\tilde{F}(B_n(x,t))$, and by Proposition 2 and (1) the mean is

$$m_n\tilde{F}(B_n(x,t)) = k_n + tk_n^{1/2} + o(k_n^{1/2}), \qquad x \in R \cap L_s$$

By using Lemma 3 and Proposition 2, we can conclude the proof in a similar manner with the out-degree case. □

Lemma 5. *Suppose $k_n \to \infty$ and (1) holds. Then*

$$\lim_{n\to\infty} \sup_{n-n^{2/3}\le l<m\le n+n^{2/3}} |k_n^{-1}E\tilde{D}^{out}_{l,n}(t)\tilde{D}^{out}_{m,n}(u) - \phi(t)\phi(u)F(L_s)^2| = 0$$

The same formula holds when replace superscripts "out" by "in".

Proof. Let $l \le m$.

For out degree, we have

$$\begin{aligned}
ED^{out}_{l,n}(t)D^{out}_{m,n}(u) &= \sum_{i=1}^{l}\sum_{j=1}^{m} P[\{\mathcal{X}_l(S_n(X_i,Y_i,t)) = k_n\} \cap \{\mathcal{X}_m(S_n(X_j,Y_j,u)) = k_n\} \\
&\quad \cap\{X_{l+1} \in S_n(X_i,Y_i,t)\} \cap \{X_{m+1} \in S_n(X_j,Y_j,u)\}] \\
&= \frac{l(l-1)}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g_{n,l,m}(x_1,y_1,x_2,y_2)F(\mathrm{d}x_1)F(\mathrm{d}x_2)\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad + \frac{l(m-l)}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g'_{n,l,m}(x_1,y_1,x_2,y_2)F(\mathrm{d}x_1)F(\mathrm{d}x_2)\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad + \frac{l}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^2} g''_{n,l,m}(x_1,y_1)F(\mathrm{d}x_1)\mathrm{d}y_1 \qquad (12)
\end{aligned}$$

where,

$$\begin{aligned}
g_{n,l,m}(x_1,y_1,x_2,y_2) &:= P[\{\mathcal{X}^{x_2}_{l-2}(S_n(x_1,y_1,t)) = k_n - 1\} \cap \{X_{l-1} \in S_n(x_1,y_1,t)\} \\
&\quad \cap\{\mathcal{X}^{x_1}_{m-2}(S_n(x_2,y_2,u)) = k_n - 1\} \cap \{X_{m-1} \in S_n(x_2,y_2,u)\}]
\end{aligned}$$

$$\begin{aligned}
g'_{n,l,m}(x_1,y_1,x_2,y_2) &:= P[\{\mathcal{X}_{l-1}(S_n(x_1,y_1,t)) = k_n - 1\} \cap \{X_l \in S_n(x_1,y_1,t)\} \\
&\quad \cap\{\mathcal{X}^{x_1}_{m-2}(S_n(x_2,y_2,u)) = k_n - 1\} \cap \{X_{m-1} \in S_n(x_2,y_2,u)\}]
\end{aligned}$$

$$\begin{aligned}
g''_{n,l,m}(x_1,y_1) &:= P[\{\mathcal{X}_{l-1}(S_n(x_1,y_1,t)) = k_n - 1\} \cap \{X_l \in S_n(x_1,y_1,t)\} \\
&\quad \cap\{\mathcal{X}_{m-1}(S_n(x_1,y_1,u)) = k_n - 1\} \cap \{X_m \in S_n(x_1,y_1,u)\}]
\end{aligned}$$

Take $x_1, x_2 \in R$, $x_1 \neq x_2$ and $y_1, y_2 \in [0, 2\pi)$. Take $\{l_n\}_{n\geq 1}$ and $\{m_n\}_{n\geq 1}$ with $n - n^{2/3} \leq l_n < m_n \leq n + n^{2/3}$. Then as $n \to \infty$,

$$\frac{n}{k_n} P(X_{l_n-1} \in S_n(x_1, y_1, t)) \to \frac{s\alpha}{2} f(x_1), \tag{13}$$

$$\frac{n}{k_n} P(X_{m_n-1} \in S_n(x_2, y_2, u)) \to \frac{s\alpha}{2} f(x_2). \tag{14}$$

Since

$$\begin{aligned} &P(\mathcal{X}^{x_1}_{m_n-2}(S_n(x_2, y_2, u)) = k_n - 1 | X_{l_n-1} \in S_n(x_1, y_1, t)) \\ &\quad \sim \frac{P(\mathcal{X}_{m_n-2}(S_n(x_2, y_2, u)) = k_n) \cdot P(X_{l_n-1} \in S_n(x_1, y_1, t))}{P(X_{l_n-1} \in S_n(x_1, y_1, t))} \sim \beta_{m_n, F(S_n(x_2,y_2,u))}(k_n), \end{aligned}$$

by Lemma 3 and (11), we obtain

$$k_n^{1/2} P(\mathcal{X}^{x_1}_{m_n-2}(S_n(x_2, y_2, u)) = k_n - 1 | X_{l_n-1} \in S_n(x_1, y_1, t)) \to \phi(u), \quad x_2 \in L_s \tag{15}$$

Let $x_1, x_2 \in \mathbb{R}^2$ with $x_2 \notin B(x_1, r_n(t) + r_n(u))$, so $B_n(x_1, t) \cap B_n(x_2, u) = \emptyset$. If $\mathcal{X}_{m_n-2}(S_n(x_2, y_2, u)) = k_n - 1$, then $\mathcal{X}_{l_n-2}(S_n(x_2, y_2, u)) = j$, for some $0 \leq j \leq k_n - 1$. Given $\mathcal{X}_{l_n-2}(S_n(x_2, y_2, u)) = j$, the conditional distribution of $\mathcal{X}_{l_n-2}(S_n(x_1, y_1, t))$ is binomial with parameter $l_n - 2 - j$ and $F(S_n(x_1, y_1, t))/(1 - F(S_n(x_2, y_2, u)))$. For all such j, if also $x_1 \in L_s$ then by (11), (1) the mean of this distribution is

$$\begin{aligned} \frac{(l_n - 2 - j)F(S_n(x_1, y_1, t))}{1 - F(S_n(x_2, y_2, u))} &= (k_n + t k_n^{1/2} + o(k_n^{1/2}))(1 + O(\frac{k_n}{n})) \\ &= k_n + t k_n^{1/2} + o(k_n^{1/2}) \end{aligned}$$

Therefore for $x_1 \in L_s$ and $x_2 \neq x_1$, by Lemma 3 we have

$$\begin{aligned} k_n^{1/2} P[\mathcal{X}^{x_2}_{l_n-2}(S_n(x_1, y_1, t)) = k_n - 1 | \{\mathcal{X}^{x_1}_{m_n-2}(S_n(x_2, y_2, u)) = k_n - 1\} \\ \cap \{X_{l_n-1} \in S_n(x_1, y_1, t)\}] \to \phi(t). \end{aligned}$$

Combining this with (13), (14) and (15), we get

$$(n^2/k_n) g_{n,l_n,m_n}(x_1, y_1, x_2, y_2) \to \phi(t)\phi(u), \quad x_1, x_2 \in R \cap L_s, x_1 \neq x_2 \tag{16}$$

On the other hand, by Chernoff bounds,

$$k_n P[\mathcal{X}^{x_1}_{m_n-2}(S_n(x_2, y_2, u)) = k_n - 1 | X_{l_n-1} \in S_n(x_1, y_1, t)] \to 0, \quad x_2 \in R \backslash L_s$$

and

$$k_n P[\mathcal{X}^{x_2}_{l_n-2}(S_n(x_1, y_1, t)) = k_n - 1] \to 0, \quad x_1 \in R \backslash L_s$$

Combing these with (13) and (14), we have

$$(n^2/k_n) g_{n,l_n,m_n}(x_1, y_1, x_2, y_2) \to 0, \quad (x_1, x_2) \in (R \times R) \backslash (L_s \times L_s) \tag{17}$$

If $x_2 \notin B(x_1, r_n(t) + r_n(u))$, setting $p_1 = F(S_n(x_2, y_2, u))$ and $p_2 = F(S_n(x_1, y_1, t))/(1 - p_1)$, we get

$$g_{n,l_n,m_n}(x_1, y_1, x_2, y_2) \leq \max_{0 \leq j \leq k_n - 1} p_1 p_2 \beta_{m_n-2, p_1}(k_n - 1) \beta_{l_n-2-j, p_2}(k_n - 1)$$

Whence by Lemma 3 and Stirling formula, there exists a constant c such that

$$g_{n,l_n,m_n}(x_1, y_1, x_2, y_2) \le c(k_n^{-\frac{1}{2}} \cdot \frac{k_n}{n})^2 = ck_n/n^2.$$

Then by (16), (17) and dominated convergence theorem, we get

$$\lim_{n\to\infty} \Big[\frac{n^2}{k_n \cdot 4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2 \setminus B(x_1, r_n(t)+r_n(u))} g_{n,l_n,m_n}(x_1, y_1, x_2, y_2) \\ F(\mathrm{d}x_2)F(\mathrm{d}x_1)\mathrm{d}y_1\mathrm{d}y_2\Big] = \phi(t)\phi(u)F(L_s)^2. \quad (18)$$

Also, by (13), (14) and (15),

$$\begin{aligned} g_{n,l_n,m_n}(x_1, y_1, x_2, y_2) &\le P(\mathcal{X}_{l_n-2}^{x_2}(S_n(x_1, y_1, t)) = k_n - 1)F(S_n(x_1, y_1, t))F(S_n(x_2, y_2, u)) \\ &= O\big(k_n^{-\frac{1}{2}} \cdot (\frac{k_n}{n})^2\big) \end{aligned}$$

Since $F(B(x_1, r_n(t) + r_n(u))) \le c \cdot (k_n/n)$ for some constant c, by (1),

$$(\frac{n^2}{k_n \cdot 4\pi^2}) \int_0^{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^2} \int_{B(x_1, r_n(t)+r_n(u))} g_{n,l_n,m_n}(x_1, y_1, x_2, y_2) \\ F(\mathrm{d}x_2)F(\mathrm{d}x_1)\mathrm{d}y_1\mathrm{d}y_2 \le c'(\frac{n^2}{k_n})(\frac{k_n}{n})(\frac{k_n^{3/2}}{n^2}) \to 0$$

Thus (18) holds with the region of integration modified to $[0, 2\pi) \times [0, 2\pi) \times \mathbb{R}^2 \times \mathbb{R}^2$. The asymptotic results for g'_{n,l_n,m_n} are just the same. Also, by similar arguments there is a constant c such that

$$l_n \cdot \sup_{\substack{x_1 \in R \\ y_1 \in [0,2\pi)}} g''_{n,l_n,m_n}(x_1, y_1) \le cnk_n^{-1/2}(k_n/n)^2 \to 0.$$

Hence (12) yields

$$k_n^{-1} ED_{l_n,n}^{out}(t)D_{m_n,n}^{out}(u) \to \phi(t)\phi(u)F(L_s)^2.$$

What remains to show is that the above formula still holds when $D_{l_n,n}^{out}$ is replaced by $\tilde{D}_{l_n,n}^{out}$; $D_{m_n,n}^{out}$ is replaced by $\tilde{D}_{m_n,n}^{out}$. We have $0 \le \hat{D}_{l_n,n}^{out}(t) \le 1$, $0 \le \hat{D}_{m_n,n}^{out}(u) \le 1$. By the proof of Lemma 4, $ED_{l_n,n}^{out}(t) = O(k_n^{1/2})$ and $ED_{m_n,n}^{out}(u) = O(k_n^{1/2})$. Therefore $E[D_{l_n,n}^{out}(t)\hat{D}_{m_n,n}^{out}(u)]$, $E[\hat{D}_{l_n,n}^{out}(t)\hat{D}_{m_n,n}^{out}(u)]$ and $E[\hat{D}_{l_n,n}^{out}(t)D_{m_n,n}^{out}(u)]$ are all $O(k_n^{1/2})$. The first part of this lemma whereby follows.

The proof for in-degree case parallels to the above approach and we leave it as an exercise for the reader.□

Lemma 6. *Suppose $k_n \to \infty$ and (1) holds. Let $t, u \in \mathbb{R}$. Then*

$$\limsup_{n\longrightarrow\infty} \big(k_n^{-3/2} \cdot \sup_{\{m||m-n|\le n^{2/3}\}} E[\tilde{D}_{m,n}^{out}(t)^2]\big) < \infty.$$

The same formula holds when replace superscript "out" by "in".

Proof. Take $\{m_n\}_{n\ge1}$ satisfying $|m_n - n| \le n^{2/3}$.

For out-degree, by (12) with $l = m = m_n$, $t = u$,

$$E[D^{out}_{m_n,n}(t)^2] = \frac{m_n(m_n-1)}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g_{n,m_n,m_n}(x_1,y_1,x_2,y_2)F(\mathrm{d}x_1)F(\mathrm{d}x_2)\mathrm{d}y_1\mathrm{d}y_2 + ED^{out}_{m_n,n}(t)$$

By Lemma 3, there is a constant c such that

$$g_{n,m_n,m_n}(x_1,y_1,x_2,y_2) \le \sup_{0<p<1}[p\max\{\beta_{m_n,p}(k_n-1),\beta_{m_n,p}(k_n-2)\}] \le (\frac{ck_n}{n})k_n^{-1/2}.$$

Also, $g_{n,m_n,m_n}(x_1,y_1,x_2,y_2) = 0$ unless $x_2 \in B(x_1, 2r_n(t))$. Whence

$$\int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g_{n,m_n,m_n}(x_1,y_1,x_2,y_2)F(\mathrm{d}x_1)F(\mathrm{d}x_2)\mathrm{d}y_1\mathrm{d}y_2 \le \frac{c'k_n^{3/2}}{n^2}$$

By Lemma 4, $ED^{out}_{m_n,n}(t) = O(k_n^{1/2})$ and $m_n(m_n-1) = O(n^2)$. So $E[D^{out}_{m_n,n}(t)^2] = O(k_n^{3/2})$. The first part of this lemma then follows, by noting $0 \le \hat{D}^{out}_{m_n,n}(t) \le 1$.

For in-degree, the same argument may be applied. Thus we conclude the proof. □

5. Proof of central limit theorems

To prove Theorem 1 and 2, we will employ useful de-Poisson techniques given in [9], [15] and later generalized in [13, 16]. We will also need Cramér-Wold device, see e.g.[8]. Now we are in position to prove our main results.

Proof of Theorem 1. Let $M \in \mathbb{N}$, $B = (b_1, \cdots, b_M) \in \mathbb{R}^M$, $T = (t_1, \cdots, t_M) \in (0,\infty)^M$.

For out-degree, $\mathcal{X} \subset \mathbb{R}^2, \mathcal{Y} \subset [0, 2\pi)$ with $\mathrm{card}(\mathcal{X}) = \mathrm{card}(\mathcal{Y})$, set

$$H_0(\mathcal{X},\mathcal{Y}) := \sum_{i=1}^{M}\sum_{(x,y)\in(\mathcal{X},\mathcal{Y})} b_i 1_{[\mathcal{X}(S(x,y,t_i^{1/2}))\ge k_n+1]}$$

and let $H_n(\mathcal{X},\mathcal{Y}) = H_0(n^{1/2}\mathcal{X},\mathcal{Y})$. $(x,y) \in \mathbb{R}^2 \times [0,2\pi) \subset \mathbb{R}^3$. Set $\xi'^{out}_n(T,B,A) := \sum_{m=1}^M b_m\xi'^{out}_n(t_m,A)$ and $\mathrm{Var}(\xi'^{out}_n(T,B,A)) := \sigma'^{out}(T,B,A)$, we have $H_n(\mathcal{P}_n,\mathcal{Y}_{N_n}) = \xi'^{out}_n(T,B,\mathbb{R}^2)$, and what's more, $(\mathcal{P}_n,\mathcal{Y}_{N_n})$ is a 3−dimensional Poisson process, which may be coupled with $(\mathcal{X}_n,\mathcal{Y}_n)$ in the same way as $\mathcal{P}_n$ does with $\mathcal{X}_n$. By Lemma 1, $n^{-1/2}(H_n(\mathcal{P}_n,\mathcal{Y}_{N_n}) - EH_n(\mathcal{P}_n,\mathcal{Y}_{N_n})) \xrightarrow{\mathrm{D}} \mathcal{N}(0,\sigma'^{out}(T,B,\mathbb{R}^2))$. Let $\mathcal{H}_\lambda$ be a 3−dimensional homogeneous Poisson process and denote point $(x,y) \in \mathbb{R}^2 \times \mathbb{R}$, $\mathcal{H}_\lambda := (\mathcal{H}^{(1)}_\lambda, \mathcal{H}^{(2)}_\lambda)$ with $x \in \mathcal{H}^{(1)}_\lambda, y \in \mathcal{H}^{(2)}_\lambda$. Next, we say $H_0(\mathcal{X},\mathcal{Y})$ is strongly stabilizing on $\mathcal{H}_\lambda$ if there are $a.s.$ finite random variables T and $\Delta(\mathcal{H}_\lambda)$ such that with probability 1, $\Delta(A) = \Delta(\mathcal{H}_\lambda)$ for all finite $A := (A_1, A_2) \subset \mathbb{R}^2 \times [0,2\pi)$ with $\mathrm{card}(A_1) = \mathrm{card}(A_2)$, satisfying $A \cap (B(0,T)\times[0,2\pi)) = \mathcal{H}_\lambda \cap (B(0,T)\times[0,2\pi))$. Here, $\triangle(\mathcal{H}_\lambda) := H_0(\mathcal{H}^0_\lambda) - H_0(\mathcal{H}_\lambda)$.

Thus, H_0 is strongly stable since it has finite range. We have

$$\begin{aligned}
E[\triangle(\mathcal{H}_\lambda)] &= E[H_0(\mathcal{H}_\lambda^0) - H_0(\mathcal{H}_\lambda)] \\
&= E\big[\sum_{i=1}^M b_i\big(\sum_{(x,y)\in\mathcal{H}_\lambda} 1_{[\mathcal{H}_\lambda^{(1),0}(S(x,y,t_i^{1/2}))\geq k+1]} + 1_{[\mathcal{H}_\lambda^{(1),0}(S(0,0,t_i^{1/2}))\geq k+1]}\big) \\
&\quad - \sum_{i=1}^M b_i\big(\sum_{(x,y)\in\mathcal{H}_\lambda} 1_{[\mathcal{H}_\lambda^{(1)}(S(x,y,t_i^{1/2}))\geq k+1]}\big)\big] \\
&= E\sum_{i=1}^M b_i\big(1_{[Poi(2\pi\lambda\cdot\frac{\alpha t_i}{2})\geq k]} + \sum_{\substack{(x,y)\in\mathcal{H}_\lambda \\ 0\in S(x,y,t_i^{1/2})}} 1_{[\mathcal{H}_\lambda^{(1)}(S(x,y,t_i^{1/2}))=k]}\big) \\
&= \sum_{i=1}^M b_i\big(\rho_{\lambda\pi\alpha t_i}([k,\infty)) + \lambda 2\pi\cdot\frac{\alpha t_i}{2}(k-1)\big).
\end{aligned}$$

By (2) and the Cox process $\mathcal{H}_{\varphi(X,Y)}$ with $\varphi(X,Y) := (1/2\pi)f(X)$, we have $E[\triangle(\mathcal{H}_{\varphi(X,Y)})]$ $= \sum_{i=1}^M b_i h(t_i)$. Set $t_{\max} = \max\{t_1,\cdots,t_M\}$, we have $|H_n(\mathcal{X}_m,\mathcal{Y}_m)| \leq m\sum_{i=1}^M |b_i|$ and

$$|H_n(\mathcal{X}_{m+1},\mathcal{Y}_{m+1}) - H_n(\mathcal{X}_m,\mathcal{Y}_m)| \leq \big(\sum_{i=1}^M b_i\big)\cdot\big[\#\{X_i\in\mathcal{X}_m | X_{m+1}\in S_n(X_i,Y_i,t_{\max})\}+1\big]$$

which is stochastically dominated by $c\cdot[Bin(m, f_{\max}\pi r_n(t_{\max})^2)+1]$ having a uniformly bounded fourth moment when $m\leq 2n$. Therefore by a simple variant of Theorem 2.16([13]) to a marked point process [16] (in particular the translation-invariance of $H_0(\mathcal{X},\mathcal{Y})$ is only required for $\mathcal{X}$), $n^{-1/2}(H_n(\mathcal{X}_n,\mathcal{Y}_n) - EH_n(\mathcal{X}_n,\mathcal{Y}_n)) \xrightarrow{\mathrm{D}} \mathcal{N}(0,\tau_{out}^2)$ with $\tau_{out}^2 :=$ ${\sigma'}^{out}(T,B,\mathbb{R}^2) - (E[\triangle(\mathcal{H}_{\varphi(X,Y)})])^2$. The first part of the theorem then follows by Cramér-Wold device.

For in-degree, let $\mathcal{X}\subset\mathbb{R}^2, \mathcal{Y}\subset[0,2\pi)$ with $\mathrm{card}(\mathcal{X}) = \mathrm{card}(\mathcal{Y})$, and the elements $(x,y)\in(\mathcal{X},\mathcal{Y})$ be ordered pairs. Reset

$$H_0(\mathcal{X},\mathcal{Y}) := \sum_{i=1}^M\sum_{x'\in\mathcal{X}} b_i 1_{[\#\{x\in\mathcal{X}|x'\in S(x,y,t_i^{1/2})\}\geq k_n+1]}$$

and let $H_n(\mathcal{X},\mathcal{Y}) = H_0(n^{1/2}\mathcal{X},\mathcal{Y})$. Set $\xi_n^{\prime in}(T,B,A) := \sum_{m=1}^M b_m\xi_n^{\prime in}(t_m,A)$ and $\mathrm{Var}\big(\xi_n^{\prime in}(T,B,A)\big) := {\sigma'}^{in}(T,B,A)$, we have $H_n(\mathcal{P}_n,\mathcal{Y}_{N_n}) = \xi_n^{\prime in}(T,B,\mathbb{R}^2)$, and $(\mathcal{P}_n,\mathcal{Y}_{N_n})$ is a 3−dimensional Poisson process coupled with $(\mathcal{X}_n,\mathcal{Y}_n)$. By Lemma 1, $n^{-1/2}(H_n(\mathcal{P}_n,\mathcal{Y}_{N_n})$ $- EH_n(\mathcal{P}_n,\mathcal{Y}_{N_n})) \xrightarrow{\mathrm{D}} \mathcal{N}(0,{\sigma'}^{in}(T,B,\mathbb{R}^2))$. Also, H_0 is strongly stable. Let $\mathcal{H}_\lambda$ be a

3−dimensional homogeneous Poisson process and $\mathcal{H}_\lambda := (\mathcal{H}_\lambda^{(1)}, \mathcal{H}_\lambda^{(2)})$ as above. Then

$$\begin{aligned}
E[\triangle(\mathcal{H}_\lambda)] &= E[H_0(\mathcal{H}_\lambda^0) - H_0(\mathcal{H}_\lambda)] \\
&= E\Big[\sum_{i=1}^{M} b_i\Big(\sum_{x'\in\mathcal{H}_\lambda^{(1)}} 1_{[\#\{x\in\mathcal{H}_\lambda^{(1),0} | x'\in S(x,y,t_i^{1/2})\}\geq k+1]} + 1_{[\#\{x\in\mathcal{H}_\lambda^{(1),0} | 0\in S(x,y,t_i^{1/2})\}\geq k+1]}\Big) \\
&\quad - \sum_{i=1}^{M} b_i\Big(\sum_{x'\in\mathcal{H}_\lambda^{(1)}} 1_{[\#\{x\in\mathcal{H}_\lambda^{(1)} | x'\in S(x,y,t_i^{1/2})\}\geq k+1]}\Big)\Big] \\
&= E\sum_{i=1}^{M} b_i\Big(1_{[Poi(2\pi\lambda\cdot\frac{\alpha t_i}{2})\geq k]} + \sum_{x'\in\mathcal{H}_\lambda^{(1)}\cap S(0,0,t_i^{1/2})} 1_{[\#\{x\in\mathcal{H}_\lambda^{(1)} | x'\in S(x,y,t_i^{1/2})\}=k]}\Big) \\
&= \sum_{i=1}^{M} b_i\Big(\rho_{\lambda\pi\alpha t_i}([k,\infty)) + 2\pi\lambda\cdot\frac{\alpha t_i}{2}(k-1)\Big).
\end{aligned}$$

The remain proof is similar with the out-degree case. □

Proof of Theorem 2. Let T and $B \in \mathbb{R}^M$.

For out-degree, $\mathcal{X} \subset \mathbb{R}^2, \mathcal{Y} \subset [0, 2\pi)$ with $\mathrm{card}(\mathcal{X}) = \mathrm{card}(\mathcal{Y})$, set

$$H_n(\mathcal{X},\mathcal{Y}) := k_n^{-1/2} \sum_{i=1}^{M} \sum_{(x,y)\in(\mathcal{X},\mathcal{Y})} b_i 1_{[\mathcal{X}(S_n(x,y,t_i))\geq k_n+1]}$$

By Lemma 2, we have $H_n(\mathcal{P}_n, \mathcal{Y}_{N_n}) = k_n^{-1/2}\xi_n'^{out}(T, B, \mathbb{R}^2)$ and $n^{-1/2}(H_n(\mathcal{P}_n, \mathcal{Y}_{N_n}) - EH_n(\mathcal{P}_n, \mathcal{Y}_{N_n})) \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma'^{out}(T, B, \mathbb{R}^2))$. Set $\alpha := \sum_{i=1}^M b_i\phi(t_i)F(L_s)$, and $R_{m,n}^{out} := H_n(\mathcal{X}_{m+1}, \mathcal{Y}_{m+1}) - H_n(\mathcal{X}_m, \mathcal{Y}_m)$. Then $R_{m,n}^{out} = k_n^{-1/2}\sum_{i=1}^M b_i\tilde{D}_{m,n}^{out}(t_i)$. By Lemma 4, 5 and 6, we have

$$\lim_{n\to\infty}\Big(\sup_{n-n^{2/3}\leq m\leq n+n^{2/3}} |ER_{m,n}^{out} - \alpha|\Big) = 0$$

$$\lim_{n\to\infty}\Big(\sup_{n-n^{2/3}\leq m<m'\leq n+n^{2/3}} |E[R_{m,n}^{out}R_{m',n}^{out}] - \alpha^2|\Big) = 0$$

and

$$\lim_{n\to\infty}\Big(n^{-1/2}\sup_{n-n^{2/3}\leq m\leq n+n^{2/3}} E[(R_{m,n}^{out})^2]\Big) = 0.$$

respectively. Also $|H_n(\mathcal{X}_m, \mathcal{Y}_m)| \leq m\sum_{i=1}^M |b_i|$. Then Theorem2.12([13]) implies $n^{-1/2} \cdot (H_n(\mathcal{X}_n, \mathcal{Y}_n) - EH_n(\mathcal{X}_n, \mathcal{Y}_n)) \xrightarrow{\mathrm{D}} \mathcal{N}(0, \sigma^{out}(T, B))$, with $\sigma^{out}(T, B) := \sigma'^{out}(T, B, \mathbb{R}^2) - \alpha^2$. Hence, $\sigma^{out}(T, B) = \mathrm{Var}\sum_{i=1}^M b_i\xi_\infty^{out}(t_i)$. The first part of the theorem then follows by Cramér-Wold device.

For in-degree, let

$$H_n(\mathcal{X},\mathcal{Y}) := k_n^{-1/2} \sum_{i=1}^{M} \sum_{x'\in\mathcal{X}} b_i 1_{[\#\{x\in\mathcal{X} | x'\in S_n(x,y,t_i)\}\geq k_n+1]}$$

We then argue likewise to complete the proof. □

6. Further discussion and remarks

In the above sections, we consider $d = 2$ and Y_i uniformly distributed. A natural generalization is to consider higher dimensions. For example, for $d = 3$, instead of a sector with amplitude α, we have to consider a spherical sector $SS(X, Y, Z, r)$ which is the region bounded by a cone with vertex X, central angle α and a sphere with center X and radius r. We take X as the origin and build the standard right-handed coordinate system. Let the chief axis of the cone be a ray l, project l onto $xOy-$plane, and call it l'. Let Y be the angle between positive $x-$axis and l' and $Z + (\alpha/2)$ be the angle between l and l'. $Y, Z \in [0, 2\pi)$. Consequently, the formal definition of this "random spherical sector graph" is easily stated. If Y and Z have uniform distribution, and instead of condition (1) we assume $k_n/n^{\frac{2}{d+2}}$ tends to 0 and modify the definitions of $r_n(t)$ accordingly, then analogous results corresponding to those appeared in above sections can be derived. Actually, we have for example, $\xi_n^{out}(t, A) = \sum_{i=1}^n 1_{[\mathcal{X}_n(SS_n(X_i,Y_i,Z_i,t))\geq k_n+1]\bigcap[X_i\in A]}$. Let $p_n(x, y, z, t) = F(SS_n(x, y, z, t))$, then

$$E[\xi_n^{out}(t, A)] = \frac{n}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_A P[Bin(n-1, p_n(x, y, z, t)) \geq k_n]f(x)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Also, $\xi_n^{in}(t, A) = \sum_{i=1}^n 1_{[\#\{X_j\in\mathcal{X}_n|X_i\in SS_n(X_j,Y_j,Z_j,t)\}\geq k_n+1]\bigcap[X_i\in A]}$. Let $q_n(x, t) = F(B_n(x, t)) \cdot \left(\frac{1-\cos(\alpha/2)}{2}\right)$, then

$$E[\xi_n^{in}(t, A)] = n\int_A P[Bin(n-1, q_n(x, t)) \geq k_n]f(x)\mathrm{d}x.$$

Another direction to investigate is to consider probability density g of Y other than the uniform density. Suppose $EY < \infty$. For out-degree case, we may proceed smoothly by similar argument, whereas for in-degree the story is different. Say, we consider in-degree of a vertex u. Suppose $\|u-v\| < r$. Since the inclination of sector S_v now is not uniformly at random (as we now consider a general density g), we will have distinct thinning probability for different v. Moreover, the probability of vertex u lying in the sector S_v essentially relies on not only the distance between them but also the position of both vertices u and v. Then the computation is inevitably involved and the above de-Poisson technique is no longer valid.

We mention that the model is less interesting when using other non-Euclidean norm in application viewpoint. It is easy to see when $d = 2$, if we take l^p $(1 \leq p \leq \infty)$ norm, and $\alpha = \pi/2, \pi, 3\pi/2$ or 2π, the above results still hold, due to the symmetry of the coordinate vectors under such norm.

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